JOURNAL OF COMPUTATIONAL PHYSICS 26, 162–168 (1978)

On the Numerical Solution of a Certain Nonlinear Integro-Differential Equation

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Received July 13, 1976; revised February 23, 1977

The writers give an existence and uniqueness theorem for a nonlinear integrodifferential equation which occurs in the statistical theory of turbulent diffusion. A numerical algorithm is given and computational examples considered.

Consider the nonlinear integro-differential equation

$$u'(t) + a(t)\,u(t) + \int_0^t k(t, s)\,u(t-s)\,u(s)\,ds = f(t), \qquad 0 \leqslant t \leqslant L, \qquad u(0) = c, \tag{1}$$

where the functions $a(t)$, $f(t)$, and $k(t, s)$ are continuous for $0 \leqslant t \leqslant L$, $0 \leqslant s \leqslant L$, and c is a constant. Equations of this type occur as model equations for describing turbulent diffusion (see Velikson [5] and Monin and Yaglom [4]). In this note we prove an existence and uniqueness theorem for the equation and give a fourth-order numerical algorithm for solving it. Computational examples are also given.

Equation (1) can be transformed to an equivalent integral equation. Let

$$A(t) = \int_0^t a(s)\,ds. \tag{2}$$

Multiplying Eq. (1) by $e^{A(t)}$, we obtain

$$(e^{A(t)}u(t))' = e^{A(t)}f(t) - e^{A(t)} \int_0^t k(t, s)\,u(t-s)\,u(s)\,ds.$$

0021-9991/78/0262-0162$02.00/0

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Integrating from 0 to t and noting that $u(0) = c$, we have

$$u(t) = ce^{-A(t)} + \int_0^t e^{-[A(t)-A(\tau)]}f(\tau)\,d\tau - \int_0^t e^{-[A(t)-A(\sigma)]} \times \int_0^\sigma k(\sigma, s)\,u(\sigma - s)\,u(s)\,ds\,d\sigma, \tag{3}$$

which is equivalent to Eq. (1). Let

$$(F(u))(t) = ce^{-A(t)} + \int_0^t e^{-[A(t)-A(\tau)]}f(\tau)\,d\tau - \int_0^t e^{-[A(t)-A(\sigma)]} \int_0^\sigma k(\sigma, s)\,u(\sigma - s)\,u(s)\,ds\,d\sigma. \tag{4}$$

Then Eq. (3) can be written as

$$u(t) = (F(u))(t). \tag{5}$$

Let

$$u_0(t) = ce^{-A(t)} + \int_0^t e^{-[A(t)-A(\tau)]}f(\tau)\,d\tau \tag{6}$$

and

$$u_n(t) = (F(u_{n-1}))(t), \qquad n = 1, 2, \ldots. \tag{7}$$

Let $\| u \| = \max_{0 \leqslant t \leqslant L} | u(t)|$. We have the following theorem.

THEOREM. *Let $a(t) \geqslant 0$ for all t. If*

$$| c | + \int_0^L | f(\tau)|\,d\tau \leqslant \tfrac{1}{2} \tag{8}$$

and

$$\int_0^L \int_0^\sigma | k(\sigma, s)|\,ds\,d\sigma < \tfrac{1}{2}, \tag{9}$$

then the sequence $\{u_n\}$ determined by (6) *and* (7) *converges uniformly to a unique solution of Eq.* (1).

Proof. From (7), (4), and (6) we have

$$\begin{aligned} u_n(t) &= ce^{-A(t)} + \int_0^t e^{-[A(t)-A(\tau)]}f(\tau)\,d\tau \\ &\quad - \int_0^t e^{-[A(t)-A(\sigma)]} \int_0^\sigma k(\sigma, s)\,u_{n-1}(\sigma - s)\,u_{n-1}(s)\,ds\,d\sigma \\ &= u_0(t) - \int_0^t e^{-[A(t)-A(\sigma)]} \int_0^\sigma k(\sigma, s)\,u_{n-1}(\sigma - s)\,u_{n-1}(s)\,ds\,d\sigma. \end{aligned}$$

Since $a(t) \geqslant 0$, we have by (2), $A(t) \geqslant 0$ and increasing for all t. Thus

$$| u_n(t)| \leqslant \| u_0 \| + \| u_{n-1} \|^2 \int_0^t \int_0^\sigma | k(\sigma, s)| \, ds \, d\sigma, \qquad n = 1, 2, \ldots.$$

It follows from (8) and (9) that

$$\| u_0 \| \leqslant | c | + \int_0^L | f(\tau)| \, d\tau \leqslant \tfrac{1}{2} \tag{10}$$

and

$$\| u_n \| \leqslant \| u_0 \| + \| u_{n-1} \|^2 \int_0^L \int_0^\sigma | k(\sigma, s)| \, ds \, d\sigma < 1, \tag{11}$$

for $n = 1, 2, \ldots$.

Now, for any i and j, we have

$$\begin{aligned} &|(F(u_i))(t) - (F(u_j))(t)| \\ &\quad \leqslant \int_0^t e^{-[A(t)-A(\sigma)]} \int_0^\sigma | k(\sigma, s)| \, | u_i(\sigma - s) \, u_i(s) - u_j(\sigma - s) \, u_j(s)| \, ds \, d\sigma \\ &\quad \leqslant \int_0^t \int_0^\sigma | k(\sigma, s)| \, | u_i(\sigma - s)(u_i(s) - u_j(s)) + u_j(s)(u_i(\sigma - s) - u_j(\sigma - s))| \, ds \, d\sigma \\ &\quad \leqslant \| u_i - u_j \| (\| u_i \| + \| u_j \|) \int_0^t \int_0^\sigma | k(\sigma, s)| \, ds \, d\sigma, \end{aligned}$$

and hence

$$\| F(u_i) - F(u_j)\| \leqslant \| u_i - u_j \| (\| u_i \| + \| u_j \|) \int_0^L \int_0^\sigma | k(\sigma, s)| \, ds \, d\sigma.$$

Thus by (9), (10), and (11) there exists a positive number $\omega < 1$ such that

$$\| F(u_i) - F(u_j)\| \leqslant \omega \| u_i - u_j \|$$

for all $i, j = 0, 1, 2, \ldots$. Then by a standard argument similar to the proof of the contraction mapping theorem (see, for example, Kolmogorov and Fomin [2]), one can show that the sequence $\{u_n\}$ converges uniformly to a unique solution of Eq. (3). This completes the proof.

We next give a stepwise procedure for finding the numerical solution of Eq. (1). The method has $O(h^4)$ local truncation error. The algorithm allows us to express each u_n in terms of u_i, $i = 0, 1, \ldots, n - 1$, and thus avoid iterations. The convergence of this numerical method is justified by Mocarsky's theorem [3, p. 236].

Integrating Eq. (1) from 0 to t, we obtain

$$u(t) = u(0) - \int_0^t a(\tau) \, u(\tau) \, d\tau + \int_0^t f(\tau) \, d\tau - \int_0^t \int_0^z k(z, s) \, u(z - s) \, u(s) \, ds \, dz. \tag{12}$$

To advance from $t = 0$ to $t = h$ we approximate the integrals by Simpson's rule to obtain

$$u_1 = u_0 - (h/6)(a_0 u_0 + 4a_{1/2}u_{1/2} + a_1 u_1) + (h/6)(f_0 + 4f_{1/2} + f_1)$$
$$- (h/6)\Big[4\int_0^{h/2} k(h/2, s)\, u((h/2) - s)\, u(s)\, ds + \int_0^h k(h, s)\, u(h - s)\, u(s)\, ds\Big],$$

where $u_0 = u(0) = c$, $a_0 = a(0)$, $a_{1/2} = a(h/2)$, $a_1 = a(h)$, $f_0 = f(0)$, etc. Approximating the inner integrals by the trapezoidal rule, we have

$$\begin{aligned} u_1 &= u_0 - (h/6)(a_0 u_0 + 4a_{1/2}u_{1/2} + a_1 u_1) + (h/6)(f_0 + 4f_{1/2} + f_1) \\ &\quad - (h^2/6)[k(h/2, 0)\, u_{1/2}u_0 + k(h/2, h/2)\, u_0 u_{1/2}] \\ &\quad - (h^2/12)[k(h, 0)\, u_1 u_0 + k(h, h)\, u_0 u_1]. \end{aligned} \tag{13}$$

For the $u_{1/2}$ terms we use the following approximations in order to have $O(h^4)$ starting errors. For $u_{1/2}$ in the expression for $\int_0^h a(\tau)\, u(\tau)\, d\tau$, we apply the formula (see Hildebrand [1, p. 117, Problem 6])

$$u(x) = \frac{(x_1 - x)(x + x_1 - 2x_0)}{(x_1 - x_0)^2} u(x_0) + \frac{(x - x_0)^2}{(x_1 - x_0)^2} u(x_1)$$
$$+ \frac{(x - x_0)(x_1 - x)}{x_1 - x_0} u'(x_0) + \frac{1}{6}(x - x_0)^2 (x - x_1)\, u'''(\xi)$$

to obtain

$$u_{1/2} = \tfrac{3}{4}u_0 + \tfrac{1}{4}u_1 + (h/4)\, u'_0 + O(h^3), \tag{14}$$

where $u'_0 = u'(0) = f(0) - a(0)\, u(0) = f(0) - ca(0)$. For $u_{1/2}$ in the expression for the double integral, we approximate it by

$$u_{1/2} = u_0 + (h/2)\, u'_0 . \tag{15}$$

Substituting (14) and (15) into (13) and solving for u_1, we obtain

$$u_1 = Z_1/W_1 ,$$

where

$$\begin{aligned} Z_1 &= u_0 - (h/6)\, a_0 u_0 - (h/2)\, a_{1/2}u_0 - (h^2/6)\, a_{1/2}u'_0 \\ &\quad + (h/6)(f_0 + 4f_{1/2} + f_1) - (h^2/6)[k(h/2, 0) + k(h/2, h/2)](u_0 + (h/2)\, u'_0)\, u_0 \end{aligned}$$

and

$$W_1 = 1 + (h/6)(a_{1/2} + a_1) + (h^2/12)[k(h, 0) + k(h, h)]\, u_0 .$$

To obtain u_2 we evaluate Eq. (12) at $t = 2h$ and approximate the single integrals with Simpson's rule, the outer integral of the double integral with Simpson's rule, and the inner integral with the trapezoidal rule. We have

$$\begin{aligned} u_2 = u_0 &- (h/3)(a_0u_0 + 4a_1u_1 + a_2u_2) + (h/3)(f_0 + 4f_1 + f_2) \\ &- (2h^2/3)[k(h, 0)\, u_1u_0 + k(h, h)\, u_0u_1] \\ &- (h^2/3)[\tfrac{1}{2}k(2h, 0)\, u_2u_0 + k(2h, h)\, u_1^2 + \tfrac{1}{2}k(2h, 2h)\, u_0u_2]. \end{aligned}$$

Solving for u_2, we obtain

$$u_2 = Z_2/W_2,$$

where

$$\begin{aligned} Z_2 = u_0 &- (h/3)(a_0u_0 + 4a_1u_1) + (h/3)(f_0 + 4f_1 + f_2) \\ &- (2h^2/3)[k(h, 0) + k(h, h)]\, u_1u_0 - \tfrac{1}{3}h^2k(2h, h)\, u_1^2 \end{aligned}$$

and

$$W_2 = 1 + \tfrac{1}{3}ha_2 + \tfrac{1}{6}h^2[k(2h, 0) + k(2h, 2h)]\, u_0 .$$

To obtain u_r, $r = 3, 4,..., N$, we integrate Eq. (1) from $(r-1)h$ to rh and then approximate the resulting equation using the Adams–Moulton h^4 method. We have

$$\begin{aligned} u_r = u_{r-1} &- (h/12)(5a_ru_r + 8a_{r-1}u_{r-1} - a_{r-2}u_{r-2}) \\ &+ (h/12)(5f_r + 8f_{r-1} - f_{r-2}) - (h/12)\Big[5\int_0^{rh} k(rh, s)\, u(rh - s)\, u(s)\, ds \\ &+ 8\int_0^{(r-1)h} k((r-1)\, h, s)\, u((r-1)\, h - s)\, u(s)\, ds \\ &- \int_0^{(r-2)h} k((r-2)\, h, s)\, u((r-2)\, h - s)\, u(s)\, ds\Big] . \end{aligned}$$

We then approximate the inner integrals with the trapezoidal rule to obtain

$$\begin{aligned} u_r = u_{r-1} &- (h/12)(5a_ru_r + 8a_{r-1}u_{r-1} - a_{r-2}u_{r-2}) \\ &+ (h/12)(5f_r + 8f_{r-1} - f_{r-2}) \\ &- (5h^2/12)\Big[\tfrac{1}{2}k(rh, 0)\, u_ru_0 + \sum_{j=1}^{r-1} k(rh, jh)\, u_{r-j}u_j + \tfrac{1}{2}k(rh, rh)\, u_0u_r\Big] \\ &- (8h^2/12)\Big[\tfrac{1}{2}k((r-1)\, h, 0)\, u_{r-1}u_0 + \sum_{j=1}^{r-2} k((r-1)\, h, jh)\, u_{r-1-j}u_j \\ &+ \tfrac{1}{2}k((r-1)\, h, (r-1)\, h)\, u_0u_{r-1}\Big] + (h^2/12)\Big[\tfrac{1}{2}k((r-2)\, h, 0)\, u_{r-2}u_0 \\ &+ \sum_{j=1}^{r-3} k((r-2)\, h, jh)\, u_{r-2-j}u_j + \tfrac{1}{2}k((r-2)\, h, (r-2)\, h)\, u_0u_{r-2}\Big] . \end{aligned}$$

Solving for u_r, we have

$$u_r = Z_r/W_r, \qquad r = 3, 4, \ldots, N,$$

where

$$\begin{aligned} Z_r = {} & u_{r-1} - (h/12)(8a_{r-1}u_{r-1} - a_{r-2}u_{r-2}) + (h/12)(5f_r + 8f_{r-1} - f_{r-2}) \\ & - (5h^2/12) \sum_{j=1}^{r-1} k(rh, jh)\, u_{r-j}u_j - (8h^2/12)\Big[\tfrac{1}{2}k((r-1)h, 0)\, u_{r-1}u_0 \\ & + \sum_{j=1}^{r-2} k((r-1)h, jh)\, u_{r-1-j}u_j + \tfrac{1}{2}k((r-1)h, (r-1)h)\, u_0u_{r-1}\Big] \\ & + (h^2/12)\Big[\tfrac{1}{2}k((r-2)h, 0)\, u_{r-2}u_0 + \sum_{j=1}^{r-3} k((r-2)h, jh)\, u_{r-2-j}u_j \\ & + \tfrac{1}{2}k((r-2)h, (r-2)h)\, u_0u_{r-2}\Big] \end{aligned}$$

and

$$W_r = 1 + (5h/12)\, a_r + (5h^2/24)[k(rh, 0) + k(rh, rh)]\, u_0 .$$

We consider the following computational examples.

EXAMPLE 1

$$u'(t) + \tfrac{1}{8}e^{-2t}u(t) + \int_0^t \tfrac{1}{2}e^{-(t+s)}u(t-s)\, u(s)\, ds = -\tfrac{1}{4}e^{-t} + \frac{1}{32}\, e^{-2t}, \qquad 0 \leqslant t \leqslant 10,$$

$$u(0) = \tfrac{1}{4}.$$

The exact solution is $u(t) = \frac{1}{4}e^{-t}$.

EXAMPLE 2

$$\begin{aligned} u'(t) - \frac{2}{t+1}\, u(t) + \int_0^t \frac{1}{(t+1)^2\,(s+1)^2}\, u(t-s)\, u(s)\, ds \\ = \frac{1}{48}\left[t + 1 - \frac{1}{(t+1)^2}\right], \qquad 0 \leqslant t \leqslant 4, \end{aligned}$$

$$u(0) = \tfrac{1}{4}$$

The exact solution is $u(t) = \frac{1}{4}(t+1)^2$.

Note that the conditions of our theorem are all satisfied. The approximate solutions are computed using the above algorithm. We list in Tables I and II the resulting errors. By error we mean

$$\text{error} = |\ \text{exact value} - \text{approximate value}\ |.$$

The programs are written in FORTRAN in double precision for the IBM 370/158 computer at The Cleveland State University.

TABLE I
Errors for Example 1

t	$h = 0.1$	$h = 0.05$	$h = 0.025$
0.025			4.18×10^{-10}
0.05		6.60×10^{-9}	1.90×10^{-9}
0.1	1.03×10^{-7}	2.84×10^{-8}	6.95×10^{-10}
0.2	4.01×10^{-7}	1.94×10^{-8}	1.77×10^{-9}
0.3	9.46×10^{-8}	3.56×10^{-8}	1.06×10^{-8}
0.4	4.74×10^{-7}	3.36×10^{-8}	2.27×10^{-8}
0.6	1.03×10^{-6}	7.08×10^{-9}	4.92×10^{-8}
0.8	1.45×10^{-6}	2.25×10^{-8}	7.26×10^{-8}
1.0	1.80×10^{-6}	4.22×10^{-8}	9.05×10^{-8}
2.0	3.15×10^{-6}	1.12×10^{-8}	1.19×10^{-7}
4.0	4.17×10^{-6}	9.08×10^{-8}	1.12×10^{-7}
6.0	4.33×10^{-6}	1.11×10^{-7}	1.10×10^{-7}
8.0	4.35×10^{-6}	1.13×10^{-7}	1.09×10^{-7}
10.0	4.36×10^{-6}	1.14×10^{-7}	1.09×10^{-7}

TABLE II
Errors for Example 2

t	$h = 0.1$	$h = 0.05$	$h = 0.025$
0.025			1.00×10^{-9}
0.05		1.57×10^{-8}	7.68×10^{-9}
0.1	2.43×10^{-7}	1.16×10^{-7}	3.04×10^{-8}
0.2	1.66×10^{-6}	4.56×10^{-7}	1.15×10^{-7}
0.3	3.75×10^{-6}	9.81×10^{-7}	2.47×10^{-7}
0.4	6.48×10^{-6}	1.67×10^{-6}	4.20×10^{-7}
0.6	1.37×10^{-5}	3.48×10^{-6}	8.71×10^{-7}
0.8	2.28×10^{-5}	5.78×10^{-6}	1.45×10^{-6}
1.0	3.37×10^{-5}	8.52×10^{-6}	2.13×10^{-6}
2.0	1.10×10^{-4}	2.77×10^{-5}	6.94×10^{-6}
3.0	2.18×10^{-4}	5.49×10^{-5}	1.37×10^{-5}
4.0	3.55×10^{-4}	8.92×10^{-5}	2.23×10^{-5}

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